On the Density of Iterated Line Segment Intersections

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Abstract

Given S_1, a finite set of points in the plane, we define a sequence of point sets S_i as follows: With S_i already determined, let L_i be the set of all the line segments connecting pairs of points of $\bigcup_{j=1}^{i} S_j$, and let S_{i+1} be the set of intersection points of those line segments in L_i, which cross but do not overlap. We show that with the exception of some starting configurations the set of all crossing points $\bigcup_{i=1}^{\infty} S_i$ is dense in a particular subset of the plane with nonempty interior. This region is the intersection of all closed half planes which contain all but at most one point from S_1.

Keywords: discrete geometry, computational geometry, density, intersections, line segments

1 Introduction

Given $S = S_1$, a finite set of points in the Euclidean plane, let L_1 denote the set of line segments connecting pairs of points from S_1. Next, let S_2 be the set of all the intersection points of those line segments in L_1 which do not overlap. We continue to define sets of line segments L_i and point sets S_i inductively by

$$
L_i \quad := \quad \left\{ pq \ \middle| \ p, q \in \bigcup_{j=1}^{i} S_j \ \wedge \ p \neq q \right\},
$$
$$
S_{i+1} \quad := \quad \{ x \mid \{x\} = l \cap l' \text{ where } l, l' \in L_i \} .
$$

Finally, let $S_\infty := \bigcup_{i=1}^{\infty} S_i$ denote the limit set.

In this article we show which starting configurations S give rise to density of S_∞. And for those, we prove that S_∞ is dense in a certain region $K(S)$.

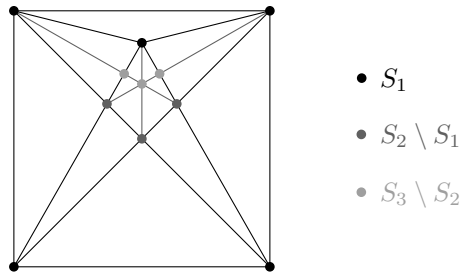
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Figure 1: The first iterations S_1, S_2 and S_3 of segment intersections.

Several results concerning the density of similar iterated constructions are known. Bezdek and Pach [2] studied the following problem: Let $S_1 = \{O_1, O_2\}$ consist of two distinct points in the plane with distance less than 2. Define S_{i+1} inductively as the set of all intersection points between the unit circles, whose centers are in S_i. They proved, if the distance between O_1 and O_2 does not equal 1 nor $\sqrt{3}$, the limit point set $\bigcup_{i=1}^{\infty} S_i$ is everywhere dense in the plane.

Kazdan [8] considered two rigid motions A and B applied to an arbitrary point p in the plane. An implication of his more powerful result is that the set $S_\infty := \bigcup_{i=1}^{\infty} S_i$ defined by the iterations $S_1 := \{p\}$ and $S_{i+1} := \{Ax, A^{-1}x, Bx, B^{-1}x | x \in S_i\}$ is everywhere dense in the plane if A is a rotation through an angle incommensurable with 2π and A and B do not commute.

Bárány, Frankl, and Maehara [1] showed that with the exception of four special cases, the set of vertices of those triangles which are obtained from a particular starting triangle T by repeated edge-reflection is everywhere dense in the plane.

Ismailescu and Radoičić [7] examined a question very similar to ours. The only difference is that they considered lines instead of line segments. They proved by applying nice elementary methods that with the exception of two cases the crossing points are dense in the whole plane. Hillar and Rhea [6] independently proved the same statement with different methods.

Despite the similarity of the two settings, the methods from the analysis of iterated line intersections cannot be transferred to our case of segment intersections. The latter problem turns out to be more difficult. It has more exceptional cases, where the crossing points are not dense in any set with non-empty interior. And in non-exceptional cases the crossing points are not dense in the whole plane but only in a particular convex region $K(S)$. In fact, if we do not consider exceptional configurations, the line result is a direct consequence of the segment result presented here. If the iterated intersections of line segments are dense in a region with non-empty interior, clearly, the intersections of lines starting with the same point set are dense in the same region and thereby in the whole plane.

Our work is motivated by another interesting problem introduced recently

by Ebbers-Baumann et al. [4]; namely how to embed a given finite point set into a geometric graph of small dilation. Here a *geometric graph* is a graph in the Euclidean plane, where the vertices are points in the plane, the edges are rectifiable curves connecting the two adjacent vertices, and the edge lengths equal the lengths of the corresponding curves. Given such a geometric graph G, for any two vertices p and q we define their *vertex-to-vertex dilation* as

$$\delta_G(p,q) := \frac{|\pi(p,q)|}{|pq|},$$

where $\pi(p,q)$ is a shortest path from p to q in G and $|\,.\,|$ denotes the Euclidean length. The *dilation of G* is defined by

$$\delta(G) := \sup_{p,\,q \text{ vertices of } G,\, p \neq q} \delta_G(p,q).$$

A geometric graph G of smallest possible dilation $\delta(G) = 1$ is called *dilation-free*. We will give a list of all cases of dilation-free planar graphs in Section 2. It can also be found at Eppstein's geometry junkyard [5]. Given a point set S in the plane, the *dilation of S* is defined by

$$\Delta(S) := \inf \{\, \delta(G) \mid G = (V,E) \text{ planar graph, } S \subseteq V \text{ and } V \setminus S \text{ finite} \,\}.$$

Determining $\Delta(S)$ seems to be very difficult. The answer is even unknown if S is a set of five points placed evenly on a circle. However, Ebbers-Baumann et al. [4] were able to prove $\Delta(S) \leq 1.1247$ for every finite point set $S \subset \mathbb{R}^2$ and they showed lower bounds for some special cases.

A natural idea for embedding S in a planar graph of small dilation is to try to find a geometric graph $G = (V,E)$, $S \subseteq V$, such that $\delta_G(p,q) = 1$ for every $p,q \in V$. Now, suppose we have found such G. Obviously, for every pair $p,q \in S$, the line segment pq must be a part of G. Since G must be planar, every intersection point of these line segments must also be in V and so on.

If this iteration produces only finitely many intersection points, i.e. the set S_∞ defined at the very beginning of this article is finite, we have a planar graph $G = (V,E)$ with $S \subseteq V$, $|V \setminus S| < \infty$ and $\delta(G) = 1$; thus $\Delta(S) = 1$. This shows that $|S_\infty| < \infty$ can only hold if S is a subset of the vertices of a dilation-free planar graph. We call those point sets *exceptional configurations*. Note that $\Delta(S) = 1$ could still hold for other sets. There could be a sequence of proper geometric graphs whose dilation does not equal 1 but converges to 1.

Our main result is the following.

Theorem 1. *Let $S = S_1$ be a set of n points in the plane, which is not an exceptional configuration. Then S_∞ is dense in the region $K(S)$.*

The region $K(S)$ can be described by the following simple definition. An example is shown in Figure 2. Until we prove that S_∞ is indeed dense in $K(S)$, we call $K(S)$ the candidate.
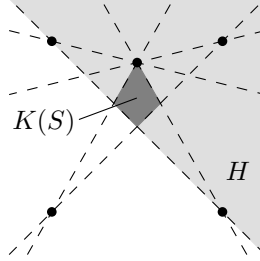
Figure 2: The candidate $K(S)$ and a closed half plane H containing all but one starting point.

Definition. The *candidate* $K(S)$ is defined as the intersection of those closed half planes which contain all the starting points except for at most one point,

$$K(S) := \bigcap_{p \in S} \bigcap_{H \supset S \setminus \{p\}} H.$$

The second intersection is taken over all closed half planes H which contain $S \setminus \{p\}$.

After the first version of our proof was worked out, Klein and Kutz [9] used the statement of Theorem 1 to prove the first non-trivial lower bound $\Delta(S) \geq 1.0000047$ which holds for every non-exceptional finite point set S. They also sketched a different proof for the special case of Theorem 1 where S consists of more than four points in convex position, not three of them on a line. The proof is based on a convergence argument similar in spirit to our Lemma 9 and on bounds to distance ratios between the converging points.

The rest of this paper is organized as follows. In Section 2 we list the exceptional configurations and we show basic properties of $K(S)$. In Section 3 we study a special case of the problem. In Section 4 we use arguments from projective geometry to prove for any non-exceptional configuration S that there exists a triangle T such that S_∞ is dense in T. Finally, in the last section we prove that in this case S_∞ is dense in $K(S)$.

2   Exceptional configurations and the candidate

Here, we list all cases of dilation-free graphs. They can also be found at Eppstein's Geometry Junkyard [5]. It can be proven by case analysis that these are all possibilities. The *exceptional configurations* are the subsets of the vertices of such graphs. Most exceptional configurations are the whole vertex set of a corresponding dilation-free graph. Only the special case described in the footnote in (iii) yields an exceptional configuration which is a proper subset.
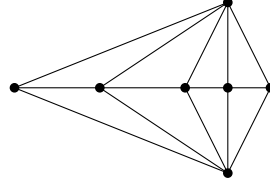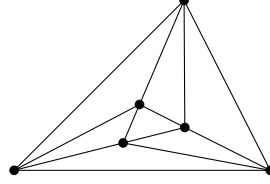
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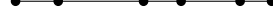
(*i*) n points on a line

(*ii*) $n-1$ points on a line, one point not on this line
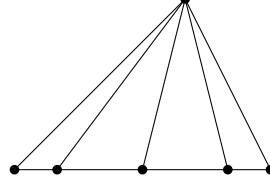
(*iii*) $n - 2$ points on a line, two points on opposite sides of this line[1]

(*iv*) a triangle (i.e. three points) nested in the interior of another triangle. Every pair of two inner points is collinear with one outer point.

The main statement we want to prove in this paper is that if S is not an exceptional configuration, the set S_∞ is dense in the candidate $K(S)$. As a first step, we list some basic properties of $K(S)$ in the following lemma. The most important one is that there cannot be any intersection point outside the candidate.

Lemma 2.

1. *If the candidate $K(S)$ is not empty, it is a convex polygon.*

2. *Every vertex of $K(S)$ belongs to $S_1 \cup S_2$.*

3. *Every intersection point lies inside of the candidate, that is $S_\infty \backslash S \subseteq K(S)$.*

Proof. 1. The convexity follows immediately from the definition because $K(S)$ is the intersection of (convex) closed half planes. The definition also implies that $K(S)$ is a subset of the convex hull ch(S). It is bounded. Furthermore, it is easy to see that in the definition of the candidate it suffices to consider only those finitely many closed half planes H which have at least two points from S on their boundary. Hence, $K(S)$ is a polygon.

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[1] Let p_1 and p_2 be the two points on opposite sides of the line, and let $p_3, \dots, p_n$ be the other points. If the segment $p_1 p_2$ intersects with the convex hull ch$(\{p_3, \dots, p_n\})$, the intersection point must be a vertex of the dilation-free graph. However, it does not have to be part of the corresponding exceptional configuration.
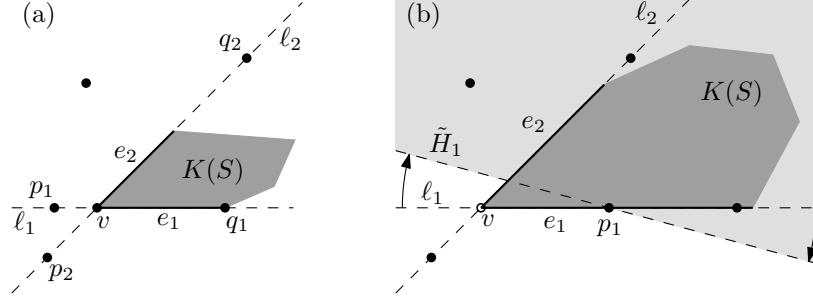
Figure 3: Every vertex v of the candidate $K(S)$ has to be an element of $S_1 \cup S_2$.

2. Consider Figure 3. Let e_1 and e_2 be two neighbor edges of the polygon $K(S)$ meeting in the vertex v, and let ℓ_1 and ℓ_2 be the lines through these edges. By the arguments above there are at least two points from S on each of the lines. If there are two points $p_1, q_1 \in S$ on ℓ_1 on opposite sides of ℓ_2, and there are two points $p_2, q_2 \in S$ on ℓ_2 on opposite sides of ℓ_1, then, as shown in Figure 3.a, the vertex v belongs to S_2 since it is the crossing point of $p_1 q_1$ and $p_2 q_2$.

Otherwise all points of $S \cap \ell_1$ lie on one side of ℓ_2 or all points of $S \cap \ell_2$ lie on one side of ℓ_1. We assume the first situation, which is depicted in Figure 3.b. Let p_1 be the point in $S \cap \ell_1$ which is closest to v. Then, if v is not a point of $S = S_1$, we can rotate the half plane H_1 belonging to e_1 around p_1 such that the turned closed half plane $\tilde{H}_1$ still contains all but one point of S and it does not contain $v \in K(S)$. This is a contradiction because by definition $K(S)$ would have to be contained in $\tilde{H}_1$.

3. Let H be a closed half plane containing $S \setminus \{p\}$. Then, clearly the crossing points of the next generation $S_2 \setminus S$ are contained in H, since all the line segments connecting points from S are either fully contained in H or they meet in p. Applying this argument inductively shows that $S_\infty \setminus S \subset H$. $\qquad\square$

3 A useful special case

In this whole section we consider a starting configuration $S = \{A, B, C, D, E, F\}$ as shown in Figure 4. The three points A, B, C are not collinear, and the three remaining points D, E, F are the midpoints of the segments AB, BC and AC respectively. We want to prove the main statement for this special case. This means, we want to show that for such a starting configuration the set of intersection points S_∞ is dense in the triangle $\triangle DEF$.

In order to do so, it is sufficient to show that S_∞ is dense on the sides of this triangle because in this case the intersections of the segments connecting the points on the sides are dense in the interior of the triangle.

Actually, for the proof we consider only a particular subset $\tilde{S}_\infty \subseteq S_\infty$. Let $\mathcal{T} := DE \cup EF \cup DF$ denote the boundary of the triangle $\triangle DEF$. Now, in
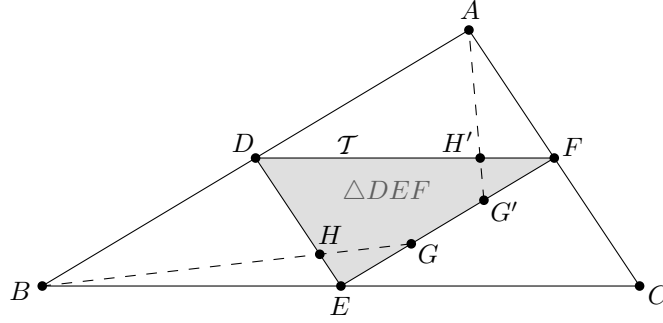
Figure 4: Proving that the intersection points are dense in the triangle $\triangle DEF$.

each step of the iteration we consider only segments which connect points on $\mathcal{T}$ with a vertex from $\{A, B, C\}$, and the intersections of such segments with $\mathcal{T}$. More formally we define

$$\tilde{S}_0 := \{D, E, F\}, \quad \forall i \in \mathbb{N}_0 : \quad \tilde{L}_i := \left\{ pq \mid p \in \tilde{S}_i,\ q \in \{A, B, C\} \right\},$$

$$\tilde{S}_{i+1} := \left\{ x \mid x \in l \cap \mathcal{T} \text{ where } l \in \tilde{L}_i \right\}, \quad \text{and} \quad \tilde{S}_\infty := \bigcup_{i=1}^{\infty} \tilde{S}_i.$$

We call a fraction $k/l \in [0, 1]$, $k, l \in \mathbb{N}_0$, $\gcd(k, l) = 1$, *constructible* if there exists a point $H \in \tilde{S}_\infty$ on DE such that $|EH|/|DE| = k/l$.

The following observation, depicted in Figure 4, contains the main step for deriving a new constructible number from an old one.

Lemma 3. *Let G be a point on the segment EF such that $|EG|/|EF| = k/l$, $k, l \in \mathbb{N}_0$. And let H be the intersection of BG and DE. Then, we have $|EH|/|DE| = k/(k + l)$.*

Proof. In the considered situation the segment BD is parallel to EF, which we denote by $BD \parallel EF$. Therefore, the triangles $\triangle EGH$ and $\triangle DBH$ are similar. This implies

$$\frac{|DH|}{|EH|} = \frac{|BD|}{|EG|} = \frac{|EF|}{|EG|} = \frac{l}{k} \tag{1}$$

The second equality holds because $DBEF$ is a parallelogram. We get

$$\frac{|DE|}{|EH|} = \frac{|DH| + |HE|}{|EH|} \overset{(1)}{=} \frac{l}{k} + 1 = \frac{l + k}{k}.$$

$\square$

We can prove this lemma analogously for all other combinations of the sides of the triangle $\mathcal{T}$ instead of EF and DE. In the next step we use this kind of symmetry to prove that if we do not use D and E in the definition of constructible

numbers but a different pair of vertices of $\triangle DEF$, the set of constructible numbers remains the same. Furthermore, we get two simple rules for deriving these numbers. Everything is summarized in the following lemma.

Lemma 4.

1. *Let $P, Q \in \{D, E, F\}$ be distinct vertices of $\triangle DEF$. Then, k/l is constructible if and only if there exists a point $H \in \tilde{S}_\infty$ on PQ such that $|PH|/|PQ| = k/l$.*

2. *The fraction k/l is constructible if and only if $1 - k/l$ is constructible.*

3. *If k/l is constructible, then, also $k/(k+l)$ is constructible.*

Proof. 1. This is a rather obvious implication from the fact that we can prove Lemma 3 also for all other possible combinations of the sides of the triangle $\triangle DEF$, not only EF and DE.

Formally, one can prove by induction on i that for every $i \in \mathbb{N}_0$ and every $k/l \in [0,1]$, $k, l \in \mathbb{N}_0$, $\gcd(k,l) = 1$, the following two statements are equivalent:

(a) There exists a point $H \in \tilde{S}_i$ on DE satisfying $|EH|/|DE| = k/l$.

(b) For every pair of distinct vertices $P, Q \in \{D, E, F\}$, there exists a point $H \in \tilde{S}_i$ on PQ satisfying $|PH|/|PQ| = k/l$.

The induction base $i = 0$ holds, because in this case (a) and (b) are both equivalent to $k/l \in \{0, 1\}$. In the induction step the implication (b) $\Rightarrow$ (a) is trivial.

Now suppose the equivalence of (a) and (b) is already shown for i. We have to prove "(a) $\Rightarrow$ (b)" for $i + 1$. Assume (a) holds for $i + 1$ and a given k/l. By definition of $\tilde{S}_{i+1}$ there exists a point $G \in \tilde{S}_i$ on EF or on DE such that $H = BG \cap DE$. Here, we consider only the case where $G \in EF$, $P = F$ and $Q = D$. All the other cases can be proved similarly. By Lemma 3 we know that $|EG|/|EF| = k/(l - k)$. Consider Figure 4. The induction hypothesis implies that there also exists a point $G' \in \tilde{S}_i$ on EF satisfying $|FG'|/|EF| = k/(l - k)$. Now we apply a variant of Lemma 3 where we replace G by G', H by H', E by F, F by E, B by A, and D remains the same. This shows that the point $H' := AG' \cap DF \in \tilde{S}_{i+1}$ satisfies $|FH'|/|DF| = k/(k + (l - k)) = k/l$.

2. This is an immediate consequence of the first statement. If we choose $P := D$ and $Q := E$, it says that there exists a point $H \in \tilde{S}_\infty$ on DE satisfying $|DH|/|DE| = k/l$, hence $|DH| = |DE|(k/l)$. This implies

$$|EH| = |DE| - |DH| = |DE|(1 - k/l).$$

3. Let k/l be constructible. Then, by choosing $P = E$ and $Q = F$ in 1., we derive that there exists a point $G \in \tilde{S}_\infty$ on EF such that $|EG|/|EF| = k/l$. We apply Lemma 3. The resulting point $H := BG \cap DE$ belongs to $\tilde{S}_\infty$ and satisfies $|EH|/|DE| = k/(k + l)$. Hence, $k/(k + l)$ is constructible. $\qquad \square$

Now we can prove the main result of this section.

Lemma 5. *Let the starting configuration $S = \{A, B, C, D, E, F\}$ be as described in the beginning of this section, cf. Figure 4. Then S_∞ is dense in $K(S) = \triangle DEF$.*

Proof. We will prove that every number in $[0,1] \cap \mathbb{Q}$ is constructible. Of course, this is a dense set in $[0,1]$. Hence, by the definition of constructible numbers and by Lemma 4.1, this implies that $\tilde{S}_\infty$ is dense in $\mathcal{T}$. Therefore, S_∞ is dense in $\mathcal{T}$, and thereby S_∞ is dense in $\triangle DEF$.

Now, we prove that every $k/l \in [0,1]$, $k, l \in \mathbb{N}_0$, $\gcd(k,l) = 1$, is constructible. We use induction on n, the sum of the numerator and the denominator of k/l.

If $n = 1$, we have $k = 0$ and $l = 1$. Therefore, $M = E$ proves the claim; the fraction 0 is constructible. Now, suppose the statement is true for the cases $k + l \leq n$. Consider an arbitrary fraction $k/l \in (0,1]$ where $k + l = n + 1$ and $\gcd(k,l) = 1$.

Case 1, $0 < k/l \leq 1/2$: We define $k' := k$ and $l' := l - k$. We then have $k', l' \in \mathbb{N}_0$, $\gcd(k', l') = 1$, $k'/l' \in (0,1]$ and $0 \leq k' + l' < k + l$. By induction hypothesis k'/l' is constructible. Lemma 4.3 implies that $k'/(k' + l') = k/l$ is constructible, too.

Case 2, $1/2 < k/l \leq 1$: We define $k' := l - k$ and $l' := l$. Then we have $k', l' \in \mathbb{N}_0$, $\gcd(k', l') = 1$, $k'/l' \in [0,1)$ and $k' + l' < k + l$. Thus by induction hypothesis k'/l' is constructible. By Lemma 4.2 we can also construct $k/l = 1 - k'/l'$. $\square$

4   Density in a triangle

In this section we want to prove that whenever S is not an exceptional configuration, the set of intersection points S_∞ is dense in a triangle. We start with a special case.
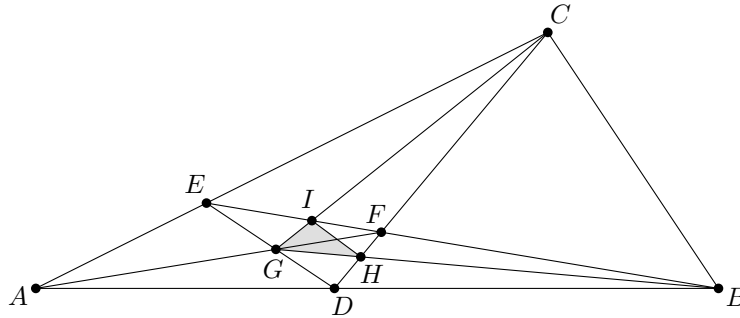

Figure 5: S_∞ is dense in $\triangle GHI$.

Lemma 6. *(Main Lemma) Let the starting configuration $S = \{A, B, C, D, E\}$ be as shown in Figure 5. The three points A, B, C are not collinear. The point D lies on the segment AB, and E is located on AC. Then S_∞ is dense in a triangle.*
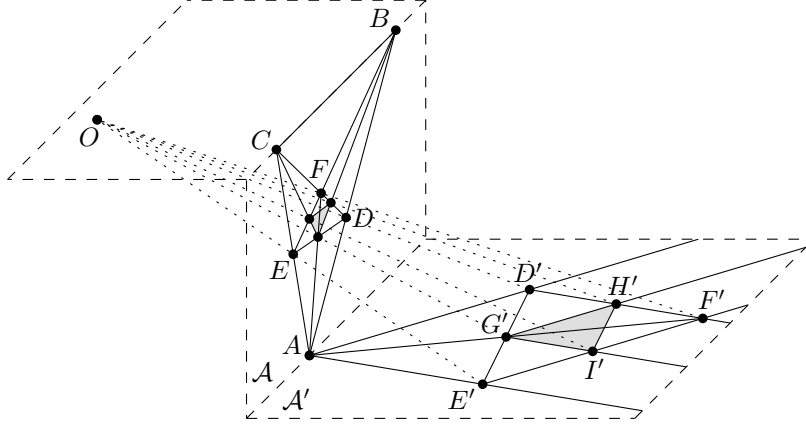

Figure 6: A useful projection of $\triangle ABC$.

Proof. We use the projection shown in Figure 6. The given two-dimensional geometric situation is represented in an affine plane $\mathcal{A}$ of $\mathbb{R}^3$ which does not contain the origin O. Additionally, we consider the affine plane $\mathcal{A}'$ which is parallel to OB and OC, and which contains A.

We can define a projection $\pi : \triangle ABC \setminus BC \to \mathcal{A}'$ by mapping each point $P \in \triangle ABC \setminus BC$ to the one point in $\mathcal{A}'$ which is hit by the line through O and P. For every such point P we denote $\pi(P)$ by P'.

Basic arguments from projective geometry show that the projection π is a homeomorphism between $\triangle ABC \setminus BC$ and $\pi(\triangle ABC \setminus BC)$, and that it maps line segments to line segments. Furthermore, the projections of two segments which meet in B are parallel in $\mathcal{A}'$, and the same holds for two segments which meet in C. For an introduction to projective geometry see for instance Bourbaki [3].

In $\mathcal{A}'$ we have $A'D' \parallel E'F'$, because the lines through AD and EF intersect in B, and analogously $A'E' \parallel D'F'$, because the corresponding lines intersect in C. Hence G' is the midpoint of $D'E'$, I' is the midpoint of $D'F'$, and H' is the midpoint of $E'F'$. By Lemma 5, for the starting configuration $S' := \{D', E', F', G', H', I'\}$ the iterated crossing points S'_∞ are dense in the triangle $\triangle G'H'I'$. Because π is a homeomorphism, this shows that S_∞ is dense in $\triangle GHI$. $\qquad\qquad\square$

Corollary 7. *Let the starting configuration S be a set of $n > 4$ points in the plane in convex position, no three of them on a line. Then S_∞ is dense in a triangle.*
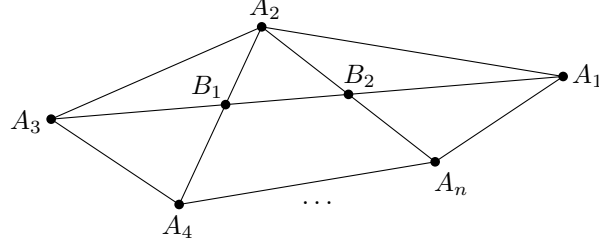

Figure 7: Proving the density in a triangle for $n > 4$ points in convex position.

Proof. Consider Figure 7. We label the starting points $A_1, A_2, \ldots, A_n$ counterclockwise. We define $B_1 := A_1A_3 \cap A_2A_4$ and $B_2 := A_1A_3 \cap A_2A_n$. Now we can apply the main lemma, Lemma 6, to the triangle $\triangle A_2A_4A_n$ with B_1 and B_2 on different sides of this triangle. Thus S_∞ is dense in a triangle. $\qquad\square$

Lemma 8. *For any non-exceptional configuration, there exists a triangle in which S_∞ is dense.*

Proof. If the convex hull of S_1, which we denote by $\mathrm{ch}(S_1)$, has more than four vertices, the claim is valid by Corollary 7. If the convex hull has three vertices A, B and C, we consider the following cases:

Case 1: There exist two additional starting points on different sides of the triangle. In this case the claim is also valid by the main lemma.
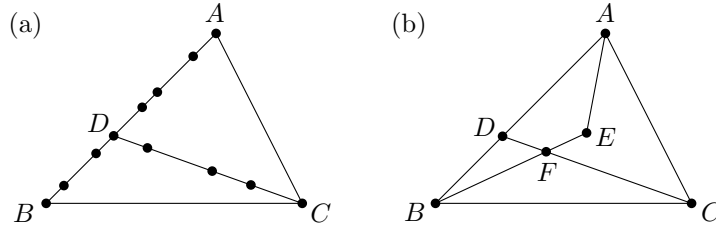
Figure 8: Case 2: $|\mathrm{ch}(S)| = 3$ and there is a starting point on one side of the triangle.

Case 2: There exists a starting point D on one side of triangle $\triangle ABC$, say on AB. The configuration contains at least one additional point, since it is not exceptional, but none on AC nor on BC. If the rest of the points $S\backslash\{A, B, C, D\}$ are all located on the line segment AB, or all on CD, then the configuration

11

belongs to the exceptional case (ii) or (iii). Hence we have one of the following cases, as shown in Figure 8.

(a) All of the additional points $S \setminus \{A, B, C, D\}$ lie on AB and CD, and on both line segments exists at least one additional point. In this case we can apply the main lemma to at least one of the triangles $\triangle ACD$ or $\triangle BCD$.

(b) There exists at least one additional point E in the interior of the triangle $\triangle ABC$, but not on CD. In this case, E is either in the interior of triangle $\triangle ACD$ or in the interior of $\triangle BCD$. Hence one of the line segments BE or AE intersects CD in a point F. Now we can apply the main lemma to triangle $\triangle ABE$ with D and F on different sides.
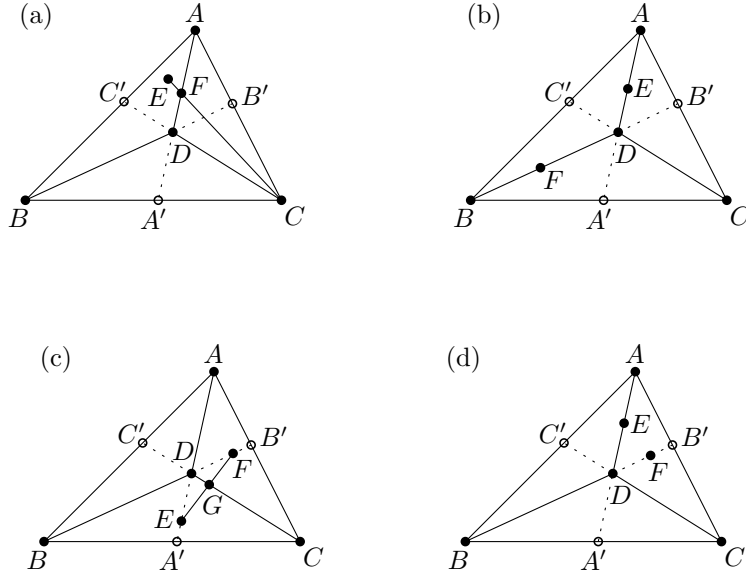
Figure 9: Case 3: $|\text{ch}(S)| = 3$ and there is no additional point on the boundary of $\triangle ABC$.

Case 3: There is no additional point on the boundary of triangle $\triangle ABC$. Hence there exists at least one point D in the interior of $\triangle ABC$. The configuration requires at least one more point not to be an exceptional configuration. Let A' be the intersection point of BC and the line through A and D. We define B' and C' analogously. Note that these points are not necessarily in S_∞!

If the rest of the points are all on AA', all on BB', or all on CC', we obtain the exceptional configuration (iii). Otherwise we distinguish the following four cases, as shown in Figure 9.

(a) There exists a point E in the interior of triangle $\triangle ABC$ which is not located on AA', BB' or CC'. Without loss of generality it is in the interior of triangle $\triangle ABD$. Thus CE intersects AD or BD in a point F; we may assume

that F lies on AD. Now we have the same situation as in Case 2.b for the triangle $\triangle ABD$ with point F on a side and point E in the interior of $\triangle ABD$. The points B, E and F cannot be collinear because C, E and F are collinear by construction and neither E nor F lies on BC.

(b) At least two of the segments AD, BD and CD contain an additional point. We may assume that there is a point E on AD and F on BD. Then we can apply the main lemma to triangle $\triangle ABD$.

(c) At least two of the segments $A'D$, $B'D$ and $C'D$ contain an additional point. We may assume that there is a point E on $A'D$ and a point F on $B'D$. Then E lies in the interior of $\triangle BCD$, and F lies in the interior of $\triangle ACD$. The segment EF intersects CD in a point G, since $A'DB'C$ is convex. Now we have the case 2.b for triangle $\triangle ACD$ with G on one side and F in the interior. If the points A, F and G were collinear, all of them had to be located on AA', since E, F and G are collinear, and A and E lie on AA'. This contradicts F being contained in the interior of $\triangle ACD$.

(d) One of the segments AD, BD or CD contains an additional point E, and one of the segments $A'D$, $B'D$ or $C'D$ contains an additional point F. We may assume $E \in AD$. Now, if we had only additional points on $A'D$, this would be exceptional case (iii). Hence we may assume $F \in B'D$. The other cases can be treated analogously. All the starting points in $S \setminus \{A, B, C, D\}$ lie on AD or $B'D$ since otherwise we had case 3.(a), (b) or (c). If we have only one additional starting point E on AD and only one point F on $B'D$, and C, E and F are collinear, then we have exceptional configuration (iv). Otherwise, there exist starting points E on AD and F on $B'D$ such that C, E and F are not collinear. We have case 2.(b) for triangle $\triangle ACD$.
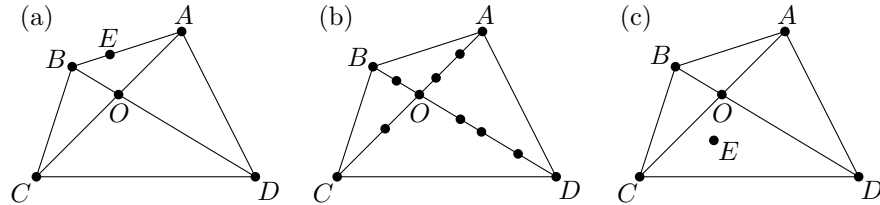
Figure 10: Subcases, if the convex hull of a non-exceptional starting configuration has four vertices.

In the remaining case the convex hull has four vertices, say A, B, C and D in counterclockwise order. Let O denote the intersection point of AC and BD. If the additional points of $S \setminus \{A, B, C, D\}$ lie all on AC or all on BD, we have exceptional case (iii). Otherwise we distinguish three cases, as shown in Figure 10.

(a) If there exists an additional point E on one of the sides of the quadrilateral $ABCD$, say on AB, we can apply the main lemma to triangle $\triangle ABC$ with E on one side and O on the other side.

13

(b) If the rest of the points are all located on AC and BD but not only on one of them, then we can also use the main lemma for at least one of the triangles $\triangle ABO, \triangle ADO, \triangle BCO$ or $\triangle CDO$.

(c) If there is a point E in the interior of the quadrilateral but not on AC or BD, then E lies in one of the triangles $\triangle ABC$ or $\triangle ACD$. This is case 2.b for the triangle containing E. $\qquad\square$

5   Density in the candidate

We now know that, if S is not an exceptional configuration, the intersection points S_∞ are dense in a triangle. We still want to show that they are dense in $K(S)$. To this end, we introduce some lemmata. The situation of Lemma 9 is shown in Figure 11.
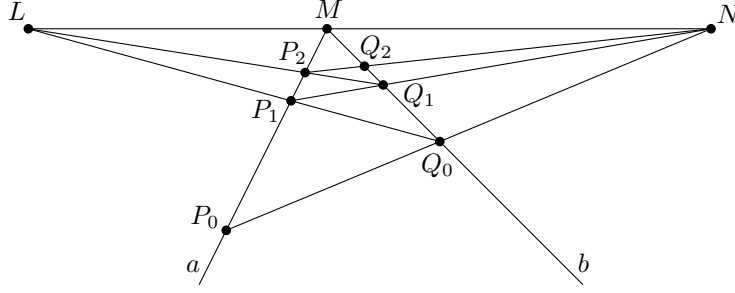

Figure 11: The point sequences $(P_i)_{i\in\mathbb{N}_0}$ and $(Q_i)_{i\in\mathbb{N}_0}$ converge to M.

Lemma 9. *Let L, M and N be three distinct points such that M lies on the line segment LN, and let a and b be two rays emanating from M on the same side of LM. Let P_0 be a point on the ray a. We define a sequence of points inductively by $Q_i := P_iN \cap b$, $P_{i+1} := Q_iL \cap a$ for every $i \in \mathbb{N}_0$. Then the point sequence $(P_i)_{i\in\mathbb{N}_0}$ converges to M on a and the point sequence $(Q_i)_{i\in\mathbb{N}_0}$ converges to M on b.*

Proof. One way to prove this simple result is to apply the same trick as in the proof of the main lemma. We use the projection shown in Figure 12. The given two-dimensional geometric situation is represented in an affine plane $\mathcal{A}$ of $\mathbb{R}^3$ which does not contain the origin O. Additionally, we consider an affine plane $\mathcal{A}'$ which is parallel to OL and ON, and which contains P_0.

We define a projection $\pi : \triangle P_0LN \setminus LN \to \mathcal{A}'$ by mapping each point $P \in \triangle P_0LN \setminus LN$ to the one point in $\mathcal{A}'$ which is hit by the line through O and P. Again for every such point P we denote $\pi(P)$ by P'.

And basic arguments from projective geometry show that the projection π is injective and maps line segments to line segments. Furthermore, the projections

Figure 12: Proving Lemma 9 with a projection.

of two segments which meet in M are parallel in $\mathcal{A}'$, and the same holds for two segments which meet in L or N.

Because of this, we know that the projected rays $a' := \pi(a)$ and $b' := \pi(b)$ are parallel in $\mathcal{A}'$. With the same argument we get

$$P_0'Q_0' \parallel P_1'Q_1' \parallel \cdots \parallel P_i'Q_i' \parallel \cdots \text{ and } Q_0'P_1' \parallel Q_1'P_2' \parallel \cdots \parallel Q_i'P_{i+1}' \parallel \cdots.$$

Hence $P_i'P_{i+1}'Q_{i+1}'Q_i'$ and $Q_i'Q_{i+1}'P_{i+2}'P_{i+1}'$ are parallelograms for every $i \in \mathbb{N}_0$. This shows

$$|P_0'P_1'| = |Q_0'Q_1'| = |P_1'P_2'| = |Q_1'Q_2'| = \ldots.$$

These equations imply that the sequences $(P_i')_{i \in \mathbb{N}_0}$ and $(Q_i')_{i \in \mathbb{N}_0}$ diverge, they converge to the endpoint at infinity of a', b' respectively. This shows that $(P_i)_{i \in \mathbb{N}_0}$ converges to M, because otherwise it had to converge to a different point on a, but then $(P_i')_{i \in \mathbb{N}_0}$ had to converge to the corresponding point on a'. Analogously we derive that $(Q_i)_{i \in \mathbb{N}_0}$ converges to M. $\qquad\square$

We can generalize this statement to the situation shown in Figure 13. For two distinct points P, Q, let line(P, Q) denote the line which passes through both points.

Corollary 10. *Let L and N be two points, and this time let M be a point not on the segment LN and not collinear with L and N. Let H_L be the half plane bounded by* line(L, M) *which does not contain N, and similarly let H_N be the half plane bounded by* line(M, N) *which does not contain L. Let a and b be two rays emanating from M and lying in the interior of $H_L \cap H_N$. If P_0 is a point on the ray a and we define point sequences $(P_i)_{i \in \mathbb{N}_0}$ and $(Q_i)_{i \in \mathbb{N}_0}$ analogously to Lemma 9, then both sequences converge to M.*
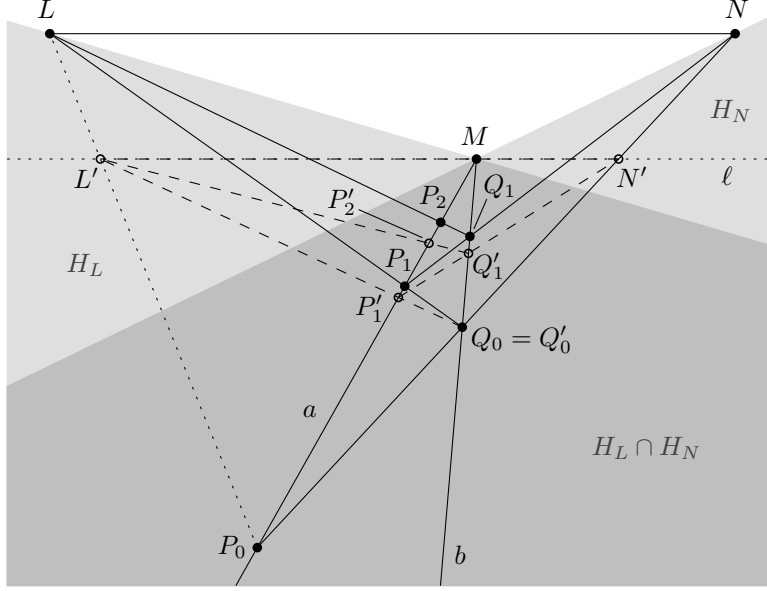
Figure 13: These point sequences $(P_i)_{i \in \mathbb{N}_0}$ and $(Q_i)_{i \in \mathbb{N}_0}$ also converge to M.

Proof. Consider Figure 13. Let ℓ be the line through M which is parallel to line(L, N). Then we define $L' := \ell \cap P_0 L$ and $N' := \ell \cap P_0 N$. Now, because M lies on the segment $L'N'$, we can apply Lemma 9 to L', M, N', a, b and $P_0' := P_0$ to construct two convergent sequences, $(P_i')_{i \in \mathbb{N}_0}$ on a and $(Q_i')_{i \in \mathbb{N}_0}$ on b, which 'push' $(P_i)_{i \in \mathbb{N}_0}$ and $(Q_i)_{i \in \mathbb{N}_0}$ towards M, that is $|P_i M| \leq |P_i' M| \searrow 0$ and $|Q_i M| \leq |Q_i' M| \searrow 0$. $\qquad\square$

The last technical tool we need is the following fact, depicted in Figure 14. We omit the simple proof.
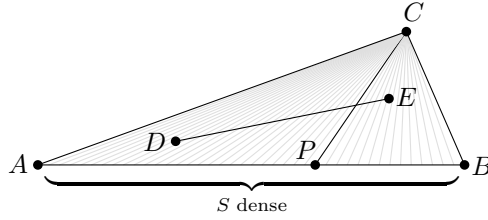

Figure 14: If S is dense on AB, the intersections $\{PC \cap DE \mid P \in S\}$ are dense on DE.

Lemma 11. *Let $\triangle ABC$ be a non-degenerate triangle, i.e. A, B and C are not collinear, and let D and E be two points in $\triangle ABC$ which are not collinear with C. If we connect every point of a point set S, which is dense on AB, with C, then the intersections of the resulting line segments with DE are dense on DE.*

Finally, before we can prove the main result, we introduce some additional useful notation.
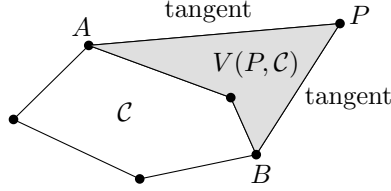

Figure 15: The visibility cone of P with respect to $\mathcal{C}$, $V(P,\mathcal{C})$.

Definition. Let $\mathcal{C}$ be a convex polygon and let P be a point outside of $\mathcal{C}$. As before, for any set $S \subset \mathbb{R}^2$, we write $\mathrm{ch}(S)$ to denote the convex hull of S. We define the *visibility cone* of P with respect to $\mathcal{C}$ by

$$V(P,\mathcal{C}) := \mathrm{ch}(\{P\}\cup\mathcal{C})\setminus\mathcal{C}.$$

We call the two edges of $V(P,\mathcal{C})$ which are adjacent to P the *tangents* of $\mathcal{C}$ through P.

Now, we are ready to prove the main result of this article. The following lemma contains the missing argument.

Lemma 12. *Let the starting configuration $S \subset \mathbb{R}^2$ be an arbitrary finite point set. Assume that S_∞ is dense in a convex polygon $\mathcal{C} \subset \mathrm{ch}(S)$ with nonempty interior. And let M be a point from S_∞ such that $M \notin \mathcal{C}$ and M is not a vertex of $\mathrm{ch}(S)$. Then S_∞ is dense in $\mathrm{ch}(\{M\}\cup\mathcal{C})$.*
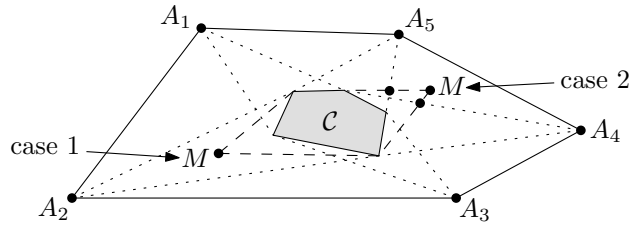

Figure 16: The two cases in the proof of Lemma 12.

Proof. Let $A_1, \ldots, A_n$ be the vertices of ch(S) in counterclockwise order. We distinguish the following two cases, as shown in Figure 16.

Case 1: If the point P lies in the interior of a visibility cone $V(A_i, \mathcal{C})$, we have $V(M, \mathcal{C}) \subset V(A_i, \mathcal{C})$. Thus by Lemma 11, S_∞ is dense on both tangents of $\mathcal{C}$ through P. Clearly, together with the density in $\mathcal{C}$, this implies that S_∞ is dense in $V(M, \mathcal{C})$, and therefore it is dense in ch($\{M\} \cup \mathcal{C}$).

If M lies on the boundary of a visibility cone $V(A_i, \mathcal{C})$, Lemma 11 shows only that S_∞ is dense on the one tangent which is not collinear with A_i. But in this special case this is enough to prove the density in ch($\{M\} \cup \mathcal{C}$).

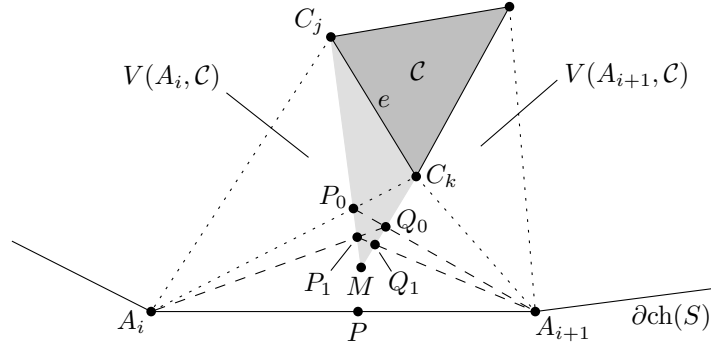Figure 17: If M does not lie inside a cone $V(A_i, \mathcal{C})$, S_∞ is dense in $V(M, \mathcal{C})$ because of Lemma 11 and Corollary 10.

Case 2: Consider Figure 17. The visibility cone $V(M, \mathcal{C})$ intersects at least one $V(A_i, \mathcal{C})$. This can be seen by considering a point P continuously moving along the boundary of ch(S). At some point its visibility cone $V(P, \mathcal{C})$ contains M. If P is a vertex of ch(S), we have case 1. Otherwise, if we define $A_{n+1} := A_1$, the point P lies on a segment $A_i A_{i+1}$, $i \in \{1, \ldots, n\}$.

The point M sees at least one whole edge e of $\mathcal{C}$. Because of $V(M, \mathcal{C}) \subseteq V(P, \mathcal{C})$, also P can see e. The edge e is also visible from A_i or from A_{i+1}. This can be proved by moving a point Q along $A_i A_{i+1}$. It cannot be that e is first invisible from Q, then becomes visible, and gets invisible again. Because e is visible from P, it therefore must also be visible from A_i or A_{i+1}. This proves that $V(M, \mathcal{C})$ intersects $V(A_i, \mathcal{C})$ or $V(A_{i+1}, \mathcal{C})$.

We assume that it intersects $V(A_i, \mathcal{C})$. And let MC_j and MC_k be the tangents of $\mathcal{C}$ through M. Then, a part of one of the tangents, say of MC_j, lies inside $V(A_i, \mathcal{C})$. Let P_0 be the intersection point of MC_j and the boundary of $V(A_i, \mathcal{C})$. Then, by Lemma 11 the set S_∞ is dense on $P_0 C_j$. Let Q_0 be the intersection of MC_k and $A_{i+1} P_0$. Then, by Lemma 11 the set S_∞ is dense on $Q_0 C_k$. We can use this argument inductively. The next step is to define $P_1 := A_i Q_0 \cap MC_j$. and $Q_1 := A_{i+1} P_1 \cap MC_k$. Lemma 11 shows that S_∞ is dense on $P_1 P_0$, and that it is dense on $Q_1 Q_0$. Corollary 10 proves that the sequences defined this way, $(P_i)_{i \in \mathbb{N}_0}$ and $(Q_i)_{i \in \mathbb{N}_0}$, both converge to M.

Hence, S_∞ is dense on both tangents, MC_j and MC_k. Because it is also dense in $\mathcal{C}$, this shows that it is dense in $\text{ch}(M \cup \mathcal{C})$. $\qquad\square$

In the end, we only have to put all the pieces together to prove that for every non-exceptional starting configuration S the set of intersection points S_∞ is dense in $K(S)$.

Proof. (of Theorem 1) If S is not an exceptional configuration, by Lemma 8 we know that S_∞ is dense in a triangle T. This implies by Lemma 2.3 that $T \subseteq K(S)$. We also know by Lemma 2 that $K(S)$ is a convex polygon whose vertices belong to S_∞. Let $P_1, P_2, \ldots, P_n$ be the vertices of $K(S)$. Then, by using Lemma 12, we can show that S_∞ is dense in $\text{ch}(T \cup \{P_1\})$. By applying Lemma 12 again, we get that S_∞ is dense in $\text{ch}(T \cup \{P_1, P_2\})$. We repeat this argument n times to prove that S_∞ is dense in $\text{ch}(T \cup \{P_1, \ldots, P_n\}) = K(S)$. $\qquad\square$

6 Acknowledgements

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